

ON NEW GENERAL INTEGRAL INEQUALITIES FOR h -CONVEX FUNCTIONS

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ABSTRACT. In this paper, we derive new estimates for the remainder term of the midpoint, trapezoid, and Simpson formulae for functions whose derivatives in absolute value at certain power are h -convex and we point out the results for some special classes of functions. Some applications to special means of real numbers are also given.

1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

holds. This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions. See [2]-[7],[11]-[14],[16],[22], the results of the generalization, improvement and extension of the famous integral inequality (1.1).

In the paper [22] a large class of non-negative functions, the so-called h -convex functions is considered. This class contains several well-known classes of functions such as non-negative convex functions, s -convex in the second sense, Godunova Levin functions and P -functions. Let us recall definitions of these special classes of functions.

Definition 1. $f : I \rightarrow \mathbb{R}$ is a Godunova-Levin function or that f belongs to the class $Q(I)$ if f is non-negative and for all $x, y \in I$ and $\alpha \in (0, 1)$ we have

$$f(\alpha x + (1-\alpha)y) \leq \frac{f(x)}{\alpha} + \frac{f(y)}{1-\alpha}$$

The class $Q(I)$ was firstly described in [8] by Godunova and Levin. Some further properties of it are given in [7, 14, 15]. Among others, it is noted that non-negative monotone and non-negative convex functions belong to this class of functions.

In 1978, Breckner introduced s -convex functions as a generalization of convex functions as follows [4]:

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Definition 2. Let $s \in (0, 1]$ be a fixed real number. A function $f : [0, \infty) \rightarrow [0, \infty)$ is said to be s -convex (in the second sense), or that f belongs to the class K_s^2 , if

$$f(\alpha x + (1 - \alpha)y) \leq \alpha^s f(x) + (1 - \alpha)^s f(y)$$

for all $x, y \in [0, \infty)$ and $\alpha \in [0, 1]$.

Of course, s -convexity means just convexity when $s = 1$. In [7] Dragomir et al. defined the concept of P -function as the following:

Definition 3. We say that $f : I \rightarrow \mathbb{R}$ is a P -function, or that f belongs to the class $P(I)$, if f is a non-negative function and for all $x, y \in I$, $\alpha \in [0, 1]$, we have

$$f(\alpha x + (1 - \alpha)y) \leq f(x) + f(y).$$

Let I and J be intervals in \mathbb{R} , $(0, 1) \subseteq J$ and h and f be real non-negative functions defined on J and I , respectively. In [22], Varošanec defined the concept of h -convexity as follows:

Definition 4. Let $h : J \rightarrow \mathbb{R}$ be a non-negative function, $h \neq 0$. We say that $f : I \rightarrow \mathbb{R}$ is an h -convex function or that f belongs to the class $SX(h, I)$, if f is non-negative function and for all $x, y \in I$ and $\alpha \in (0, 1)$ we have

$$(1.2) \quad f(\alpha x + (1 - \alpha)y) \leq h(\alpha)f(x) + h(1 - \alpha)f(y).$$

If inequality (1.2) is reversed, then f is said to be h -concave, i.e. $f \in SV(h, I)$. The notion of h -convexity unifies and generalizes the known classes of functions, s -convex functions, Gudunova-Levin functions and P -functions, which are obtained by putting in (1.2), $h(t) = t$, $h(t) = t^s$, $h(t) = \frac{1}{t}$, and $h(t) = 1$, respectively.

In [5], Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for s -convex functions in the second sense.

Theorem 1. Suppose that $f : [0, \infty) \rightarrow [0, \infty)$ is an s -convex function in the second sense, where $s \in (0, 1)$, and let $a, b \in [0, \infty)$, $a < b$. If $f \in L[a, b]$ then the following inequalities hold

$$(1.3) \quad 2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{s+1}.$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.3).

Theorem 2. Let $f \in Q(I)$, $a, b \in I$ with $a < b$ and $f \in L[a, b]$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{4}{b-a} \int_a^b f(x) dx.$$

Theorem 3. Let $f \in P(I)$, $a, b \in I$ with $a < b$ and $f \in L[a, b]$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a} \int_a^b f(x) dx \leq 2[f(a) + f(b)].$$

In [8], Dragomir et. al. proved two inequalities of Hadamard type for classes of Gudunova-Levin functions and P -functions.

In [18], Sarikaya et. al. established a new Hadamard-type inequality for h -convex functions.

Theorem 4. Let $f \in SX(h, I)$, $a, b \in I$ with $a < b$ and $f \in L([a, b])$. Then

$$(1.4) \quad \frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq [f(a) + f(b)] \int_0^1 h(t) dt.$$

The following inequality is well known in the literature as Simpson's inequality .

Let $f : [a, b] \rightarrow \mathbb{R}$ be a four times continuously differentiable mapping on (a, b) and $\|f^{(4)}\|_\infty = \sup_{x \in (a, b)} |f^{(4)}(x)| < \infty$. Then the following inequality holds:

$$\left| \frac{1}{3} \left[\frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{1}{2880} \|f^{(4)}\|_\infty (b-a)^2.$$

In recent years many authors have studied error estimations for Simpson's inequality; for refinements, counterparts, generalizations and new Simpson's type inequalities, see [1, 17, 19, 20].

In [10], Iscan obtained a new generalization of some integral inequalities for differentiable convex mapping which are connected Simpson's, midpoint and trapezoid inequalities, and he used the following lemma to prove this.

Lemma 1. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$ and $\alpha, \lambda \in [0, 1]$. Then the following equality holds:

$$\begin{aligned} & \lambda(\alpha f(a) + (1-\alpha)f(b)) + (1-\lambda)f(\alpha a + (1-\alpha)b) - \frac{1}{b-a} \int_a^b f(x) dx \\ &= (b-a) \left[\int_0^{1-\alpha} (t-\alpha\lambda) f'(tb + (1-t)a) dt \right. \\ & \quad \left. + \int_{1-\alpha}^1 (t-1+\lambda(1-\alpha)) f'(tb + (1-t)a) dt \right]. \end{aligned}$$

The main inequality in [10], pointed out, is as follows.

Theorem 5. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I^\circ$ with $a < b$ and $\alpha, \lambda \in [0, 1]$. If $|f'|^q$ is convex on $[a, b]$,

$q \geq 1$, then the following inequality holds:

$$(1.5) \left| \lambda (\alpha f(a) + (1 - \alpha) f(b)) + (1 - \lambda) f(\alpha a + (1 - \alpha) b) - \frac{1}{b - a} \int_a^b f(x) dx \right|$$

$$\leq \begin{cases} (b - a) \left\{ \gamma_2^{1-\frac{1}{q}} (\mu_1 |f'(b)|^q + \mu_2 |f'(a)|^q)^{\frac{1}{q}} \right. \\ \quad \left. + v_2^{1-\frac{1}{q}} (\eta_3 |f'(b)|^q + \eta_4 |f'(a)|^q)^{\frac{1}{q}} \right\}, & \alpha\lambda \leq 1 - \alpha \leq 1 - \lambda(1 - \alpha) \\ (b - a) \left\{ \gamma_2^{1-\frac{1}{q}} (\mu_1 |f'(b)|^q + \mu_2 |f'(a)|^q)^{\frac{1}{q}} \right. \\ \quad \left. + v_1^{1-\frac{1}{q}} (\eta_1 |f'(b)|^q + \eta_2 |f'(a)|^q)^{\frac{1}{q}} \right\}, & \alpha\lambda \leq 1 - \lambda(1 - \alpha) \leq 1 - \alpha \\ (b - a) \left\{ \gamma_1^{1-\frac{1}{q}} (\mu_3 |f'(b)|^q + \mu_4 |f'(a)|^q)^{\frac{1}{q}} \right. \\ \quad \left. + v_2^{1-\frac{1}{q}} (\eta_3 |f'(b)|^q + \eta_4 |f'(a)|^q)^{\frac{1}{q}} \right\}, & 1 - \alpha \leq \alpha\lambda \leq 1 - \lambda(1 - \alpha) \end{cases}$$

where

$$\gamma_1 = (1 - \alpha) \left[\alpha\lambda - \frac{(1 - \alpha)}{2} \right], \quad \gamma_2 = (\alpha\lambda)^2 - \gamma_1,$$

$$v_1 = \frac{1 - (1 - \alpha)^2}{2} - \alpha[1 - \lambda(1 - \alpha)],$$

$$v_2 = \frac{1 + (1 - \alpha)^2}{2} - (\lambda + 1)(1 - \alpha)[1 - \lambda(1 - \alpha)],$$

$$\mu_1 = \frac{(\alpha\lambda)^3 + (1 - \alpha)^3}{3} - \alpha\lambda \frac{(1 - \alpha)^2}{2},$$

$$\mu_2 = \frac{1 + \alpha^3 + (1 - \alpha\lambda)^3}{3} - \frac{(1 - \alpha\lambda)}{2} (1 + \alpha^2),$$

$$\mu_3 = \alpha\lambda \frac{(1 - \alpha)^2}{2} - \frac{(1 - \alpha)^3}{3},$$

$$\mu_4 = \frac{(\alpha\lambda - 1)(1 - \alpha^2)}{2} + \frac{1 - \alpha^3}{3},$$

$$\eta_1 = \frac{1 - (1 - \alpha)^3}{3} - \frac{[1 - \lambda(1 - \alpha)]}{2} \alpha(2 - \alpha),$$

$$\eta_2 = \frac{\lambda(1 - \alpha)\alpha^2}{2} - \frac{\alpha^3}{3},$$

$$\eta_3 = \frac{[1 - \lambda(1 - \alpha)]^3}{3} - \frac{[1 - \lambda(1 - \alpha)]}{2} (1 + (1 - \alpha)^2) + \frac{1 + (1 - \alpha)^3}{3},$$

$$\eta_4 = \frac{[\lambda(1 - \alpha)]^3}{3} - \frac{\lambda(1 - \alpha)\alpha^2}{2} + \frac{\alpha^3}{3}.$$

In [2] Alomari et al. obtained the following inequalities of the left-hand side of Hermite-Hadamard's inequality for s-convex mappings.

Theorem 6. Let $f : I \subseteq [0, \infty) \rightarrow \mathbb{R}$ be a differentiable mapping on I° , such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$, $q \geq 1$, is s -convex on $[a, b]$, for some fixed $s \in (0, 1]$, then the following inequality holds:

$$\begin{aligned}
 & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
 & \leq \frac{b-a}{8} \left(\frac{2}{(s+1)(s+2)} \right)^{\frac{1}{q}} \left[\{ (2^{1-s} + 1) |f'(b)|^q + 2^{1-s} |f'(a)|^q \}^{\frac{1}{q}} \right. \\
 (1.6) \quad & \left. + \{ (2^{1-s} + 1) |f'(a)|^q + 2^{1-s} |f'(b)|^q \}^{\frac{1}{q}} \right].
 \end{aligned}$$

Theorem 7. Let $f : I \subseteq [0, \infty) \rightarrow \mathbb{R}$ be a differentiable mapping on I° , such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^{\frac{p}{p-1}}$, $p > 1$, is s -convex on $[a, b]$, for some fixed $s \in (0, 1]$, then the following inequality holds:

$$\begin{aligned}
 & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \left(\frac{b-a}{4} \right) \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\frac{1}{s+1} \right)^{\frac{2}{q}} \\
 & \quad \times \left[((2^{1-s} + s + 1) |f'(a)|^q + 2^{1-s} |f'(b)|^q)^{\frac{1}{q}} \right. \\
 (1.7) \quad & \left. + ((2^{1-s} + s + 1) |f'(b)|^q + 2^{1-s} |f'(a)|^q)^{\frac{1}{q}} \right],
 \end{aligned}$$

where p is the conjugate of q , $q = p/(p-1)$.

In [19], Sarikaya et al. obtained a new upper bound for the right-hand side of Simpson's inequality for s -convex mapping as follows:

Theorem 8. Let $f : I \subseteq [0, \infty) \rightarrow \mathbb{R}$ be a differentiable mapping on I° , such that $f' \in L[a, b]$, where $a, b \in I^\circ$ with $a < b$. If $|f'|^q$, is s -convex on $[a, b]$, for some fixed $s \in (0, 1]$ and $q > 1$, then the following inequality holds:

$$\begin{aligned}
 (1.8) \quad & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{12} \left(\frac{1+2^{p+1}}{3(p+1)} \right)^{\frac{1}{p}} \\
 & \times \left\{ \left(\frac{|f'(\frac{a+b}{2})|^q + |f'(a)|^q}{s+1} \right)^{\frac{1}{q}} + \left(\frac{|f'(\frac{a+b}{2})|^q + |f'(b)|^q}{s+1} \right)^{\frac{1}{q}} \right\},
 \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

In [12], Kirmaci et al. proved the following trapezoid inequality:

Theorem 9. Let $f : I \subseteq [0, \infty) \rightarrow \mathbb{R}$ be a differentiable mapping on I° , such that $f' \in L[a, b]$, where $a, b \in I^\circ$, $a < b$. If $|f'|^q$, is s -convex on $[a, b]$, for some fixed $s \in (0, 1)$ and $q > 1$, then

$$\begin{aligned}
 (1.9) \quad & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{2} \left(\frac{q-1}{2(2q-1)} \right)^{\frac{q-1}{q}} \left(\frac{1}{s+1} \right)^{\frac{1}{q}} \\
 & \times \left\{ \left(\left| f'\left(\frac{a+b}{2}\right) \right|^q + |f'(a)|^q \right)^{\frac{1}{q}} + \left(\left| f'\left(\frac{a+b}{2}\right) \right|^q + |f'(b)|^q \right)^{\frac{1}{q}} \right\}.
 \end{aligned}$$

2. MAIN RESULTS

The following theorems give a new result of integral inequalities for h -convex functions. In the sequel of the paper I and J are intervals in \mathbb{R} , $(0, 1) \subset J$ and h and f are real non-negative functions defined on J and I , respectively and $h \in L[0, 1]$, $h \neq 0$.

Theorem 10. *Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I^\circ$ with $a < b$ and $\alpha, \lambda \in [0, 1]$. If $|f'|^q$ is h -convex on $[a, b]$, $q \geq 1$, then the following inequality holds:*

$$(2.1) \quad \left| \lambda(\alpha f(a) + (1-\alpha)f(b)) + (1-\lambda)f(\alpha a + (1-\alpha)b) - \frac{1}{b-a} \int_a^b f(x) dx \right|$$

$$\leq \begin{cases} (b-a) \left[\gamma_2^{1-\frac{1}{q}} A^{\frac{1}{q}} + v_2^{1-\frac{1}{q}} B^{\frac{1}{q}} \right] & \alpha\lambda \leq 1-\alpha \leq 1-\lambda(1-\alpha) \\ (b-a) \left[\gamma_2^{1-\frac{1}{q}} A^{\frac{1}{q}} + v_1^{1-\frac{1}{q}} B^{\frac{1}{q}} \right] & \alpha\lambda \leq 1-\lambda(1-\alpha) \leq 1-\alpha \\ (b-a) \left[\gamma_1^{1-\frac{1}{q}} A^{\frac{1}{q}} + v_2^{1-\frac{1}{q}} B^{\frac{1}{q}} \right] & 1-\alpha \leq \alpha\lambda \leq 1-\lambda(1-\alpha) \end{cases}$$

where

$$(2.2) \quad \gamma_1 = (1-\alpha) \left[\alpha\lambda - \frac{(1-\alpha)}{2} \right], \quad \gamma_2 = (\alpha\lambda)^2 - \gamma_1,$$

$$(2.3) \quad \begin{aligned} v_1 &= \frac{1-(1-\alpha)^2}{2} - \alpha[1-\lambda(1-\alpha)], \\ v_2 &= \frac{1+(1-\alpha)^2}{2} - (\lambda+1)(1-\alpha)[1-\lambda(1-\alpha)], \end{aligned}$$

$$A = |f'(b)|^q \int_0^{1-\alpha} |t-\alpha\lambda| h(t) dt + |f'(a)|^q \int_0^{1-\alpha} |t-\alpha\lambda| h(1-t) dt,$$

$$B = |f'(b)|^q \int_{1-\alpha}^1 |t-1+\lambda(1-\alpha)| h(t) dt + |f'(a)|^q \int_{1-\alpha}^1 |t-1+\lambda(1-\alpha)| h(1-t) dt$$

Proof. Suppose that $q \geq 1$. From Lemma 1 and using the well known power mean inequality, we have

$$\left| \lambda(\alpha f(a) + (1-\alpha)f(b)) + (1-\lambda)f(\alpha a + (1-\alpha)b) - \frac{1}{b-a} \int_a^b f(x) dx \right|$$

$$\leq (b-a) \left[\int_0^{1-\alpha} |t-\alpha\lambda| |f'(tb + (1-t)a)| dt + \int_{1-\alpha}^1 |t-1+\lambda(1-\alpha)| |f'(tb + (1-t)a)| dt \right]$$

$$\leq (b-a) \left\{ \left(\int_0^{1-\alpha} |t-\alpha\lambda| dt \right)^{1-\frac{1}{q}} \left(\int_0^{1-\alpha} |t-\alpha\lambda| |f'(tb + (1-t)a)|^q dt \right)^{\frac{1}{q}} \right.$$

$$(2.4) \quad + \left(\int_{1-\alpha}^1 |t-1+\lambda(1-\alpha)| dt \right)^{1-\frac{1}{q}} \left(\int_{1-\alpha}^1 |t-1+\lambda(1-\alpha)| |f'(tb+(1-t)a)|^q dt \right)^{\frac{1}{q}} \Bigg\}$$

Consider

$$I_1 = \int_0^{1-\alpha} |t-\alpha\lambda| |f'(tb+(1-t)a)|^q dt, \quad I_2 = \int_{1-\alpha}^1 |t-1+\lambda(1-\alpha)| |f'(tb+(1-t)a)|^q dt$$

Since $|f'|^q$ is h -convex on $[a, b]$,

$$(2.5) \quad I_1 \leq |f'(b)|^q \int_0^{1-\alpha} |t-\alpha\lambda| h(t) dt + |f'(a)|^q \int_0^{1-\alpha} |t-\alpha\lambda| h(1-t) dt.$$

Similarly

$$(2.6) \quad I_2 \leq |f'(b)|^q \int_{1-\alpha}^1 |t-1+\lambda(1-\alpha)| h(t) dt + |f'(a)|^q \int_{1-\alpha}^1 |t-1+\lambda(1-\alpha)| h(1-t) dt.$$

Additionally, by simple computation

$$(2.7) \quad \int_0^{1-\alpha} |t-\alpha\lambda| dt = \begin{cases} \gamma_2, & \alpha\lambda \leq 1-\alpha \\ \gamma_1, & \alpha\lambda \geq 1-\alpha \end{cases},$$

$$\gamma_1 = (1-\alpha) \left[\alpha\lambda - \frac{(1-\alpha)}{2} \right], \quad \gamma_2 = (\alpha\lambda)^2 - \gamma_1,$$

$$(2.8) \quad \int_{1-\alpha}^1 |t-1+\lambda(1-\alpha)| dt = \begin{cases} v_1, & 1-\lambda(1-\alpha) \leq 1-\alpha \\ v_2, & 1-\lambda(1-\alpha) \geq 1-\alpha \end{cases},$$

$$\begin{aligned} v_1 &= \frac{1-(1-\alpha)^2}{2} - \alpha[1-\lambda(1-\alpha)], \\ v_2 &= \frac{1+(1-\alpha)^2}{2} - (\lambda+1)(1-\alpha)[1-\lambda(1-\alpha)]. \end{aligned}$$

Thus, using (2.5) (2.8) in (2.4), we obtain the inequality (2.1). This completes the proof. \square

Corollary 1. *Under the assumptions of Theorem 10 with $q = 1$, we have*

$$\begin{aligned} & \left| \lambda(\alpha f(a) + (1-\alpha)f(b)) + (1-\lambda)f(\alpha a + (1-\alpha)b) - \frac{1}{b-a} \int_a^b f(x)dx \right| \\ & \leq (b-a) \left\{ |f'(b)| \left[\int_0^{1-\alpha} |t-\alpha\lambda| h(t)dt + \int_{1-\alpha}^1 |t-1+\lambda(1-\alpha)| h(t)dt \right] \right. \\ & \quad \left. |f'(a)| \left[\int_0^{1-\alpha} |t-\alpha\lambda| h(1-t)dt + \int_{1-\alpha}^1 |t-1+\lambda(1-\alpha)| h(1-t)dt \right] \right\}. \end{aligned}$$

Corollary 2. *Under the assumptions of Theorem 10 with $I \subseteq [0, \infty)$, $h(t) = t^s$, $s \in (0, 1]$, we have*

$$\begin{aligned} (2.9) & \left| \lambda(\alpha f(a) + (1-\alpha)f(b)) + (1-\lambda)f(\alpha a + (1-\alpha)b) - \frac{1}{b-a} \int_a^b f(x)dx \right| \\ & \leq \begin{cases} (b-a) \left[\gamma_2^{1-\frac{1}{q}} (\mu_1^* |f'(b)|^q + \mu_2^* |f'(a)|^q)^{\frac{1}{q}} \right. \\ \quad \left. + \nu_2^{1-\frac{1}{q}} (\eta_3^* |f'(b)|^q + \eta_4^* |f'(a)|^q)^{\frac{1}{q}} \right], & \alpha\lambda \leq 1-\alpha \leq 1-\lambda(1-\alpha) \\ (b-a) \left[\gamma_2^{1-\frac{1}{q}} (\mu_1^* |f'(b)|^q + \mu_2^* |f'(a)|^q)^{\frac{1}{q}} \right. \\ \quad \left. + \nu_2^{1-\frac{1}{q}} (\eta_1^* |f'(b)|^q + \eta_2^* |f'(a)|^q)^{\frac{1}{q}} \right], & \alpha\lambda \leq 1-\lambda(1-\alpha) \leq 1-\alpha, \\ (b-a) \left[\gamma_2^{1-\frac{1}{q}} (\mu_3^* |f'(b)|^q + \mu_4^* |f'(a)|^q)^{\frac{1}{q}} \right. \\ \quad \left. + \nu_2^{1-\frac{1}{q}} (\eta_3^* |f'(b)|^q + \eta_4^* |f'(a)|^q)^{\frac{1}{q}} \right], & 1-\alpha \leq \alpha\lambda \leq 1-\lambda(1-\alpha) \end{cases} \end{aligned}$$

where γ_1 , γ_2 , ν_1 and ν_2 are defined as in (2.2)-(2.3) and

$$\begin{aligned} \mu_1^* &= (\alpha\lambda)^{s+2} \frac{2}{(s+1)(s+2)} - (\alpha\lambda) \frac{(1-\alpha)^{s+1}}{s+1} + \frac{(1-\alpha)^{s+2}}{s+2}, \\ \mu_2^* &= (1-\alpha\lambda)^{s+2} \frac{2}{(s+1)(s+2)} - \frac{(1-\alpha\lambda)(1+\alpha^{s+1})}{s+1} + \frac{1+\alpha^{s+2}}{s+2}, \\ \mu_3^* &= (\alpha\lambda) \frac{(1-\alpha)^{s+1}}{s+1} - \frac{(1-\alpha)^{s+2}}{s+2}, \\ \mu_4^* &= \frac{(\alpha\lambda-1)(1-\alpha^{s+1})}{s+1} + \frac{1-\alpha^{s+2}}{s+2}, \end{aligned}$$

$$\begin{aligned}
 \eta_1^* &= \frac{1 - (1 - \alpha)^{s+2}}{s+2} - \frac{[1 - \lambda(1 - \alpha)]}{s+1} [1 - (1 - \alpha)^{s+1}], \\
 \eta_2^* &= \frac{\lambda(1 - \alpha)\alpha^{s+1}}{s+1} - \frac{\alpha^{s+2}}{s+2}, \\
 \eta_3^* &= \frac{2[1 - \lambda(1 - \alpha)]^{s+2}}{(s+1)(s+2)} - \frac{[1 + (1 - \alpha)^{s+1}][1 - \lambda(1 - \alpha)]}{s+1} + \frac{1 + (1 - \alpha)^{s+2}}{s+2}, \\
 \eta_4^* &= [\lambda(1 - \alpha)]^{s+2} \frac{2}{(s+1)(s+2)} - \lambda(1 - \alpha) \frac{\alpha^{s+1}}{s+1} + \frac{\alpha^{s+2}}{s+2}.
 \end{aligned}$$

Corollary 3. *Let the assumptions of Theorem 10 hold. Then for $h(t) = t$ the inequality (2.1) reduced to the inequality (1.5).*

Corollary 4. *Under the assumptions of Theorem 10 with $h(t) = 1$, we have*

$$\begin{aligned}
 &\left| \lambda(\alpha f(a) + (1 - \alpha)f(b)) + (1 - \lambda)f(\alpha a + (1 - \alpha)b) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
 &\leq (b-a) (|f'(b)|^q + |f'(a)|^q)^{\frac{1}{q}} \times \begin{cases} \gamma_2 + \nu_2 & \alpha\lambda \leq 1 - \alpha \leq 1 - \lambda(1 - \alpha) \\ \gamma_2 + \nu_1 & \alpha\lambda \leq 1 - \lambda(1 - \alpha) \leq 1 - \alpha \\ \gamma_1 + \nu_2 & 1 - \alpha \leq \alpha\lambda \leq 1 - \lambda(1 - \alpha) \end{cases},
 \end{aligned}$$

where $\gamma_1, \gamma_2, \nu_1$ and ν_2 are defined as in (2.2)-(2.3).

Remark 1. *In Corollary 2, if we take $\alpha = \frac{1}{2}$ and $\lambda = \frac{1}{3}$, then we have the following Simpson type inequality*

$$\begin{aligned}
 (2.10) \quad &\left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{2} \left(\frac{5}{36} \right)^{1-\frac{1}{q}} \\
 &\times \left\{ \left(\frac{(2s+1)3^{s+1} + 2}{3 \times 6^{s+1}(s+1)(s+2)} |f'(b)|^q + \frac{2 \times 5^{s+2} + (s-4)6^{s+1} - (2s+7)3^{s+1}}{3 \times 6^{s+1}(s+1)(s+2)} |f'(a)|^q \right)^{\frac{1}{q}} \right. \\
 &\left. + \left(\frac{2 \times 5^{s+2} + (s-4)6^{s+1} - (2s+7)3^{s+1}}{3 \cdot 6^{s+1}(s+1)(s+2)} |f'(b)|^q + \frac{(2s+1)3^{s+1} + 2}{3 \times 6^{s+1}(s+1)(s+2)} |f'(a)|^q \right)^{\frac{1}{q}} \right\},
 \end{aligned}$$

which is the same of the inequality in [19, Theorem 10] .

Remark 2. *In Corollary 2, if we take $\alpha = \frac{1}{2}$ and $\lambda = 0$, then we have following midpoint inequality*

$$\begin{aligned}
 (2.11) \quad &\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{8} \left(\frac{2}{(s+1)(s+2)} \right)^{\frac{1}{q}} \\
 &\times \left\{ \left(\frac{2^{1-s}(s+1)|f'(b)|^q}{2} + \frac{2^{1-s}(2^{s+2} - s - 3)|f'(a)|^q}{2} \right)^{\frac{1}{q}} \right. \\
 &\left. + \left(\frac{2^{1-s}(s+1)|f'(a)|^q}{2} + \frac{2^{1-s}(2^{s+2} - s - 3)|f'(b)|^q}{2} \right)^{\frac{1}{q}} \right\}.
 \end{aligned}$$

We note that the obtained midpoint inequality (2.11) is better than the inequality (1.6). Because $\frac{s+1}{2} \leq 1$ and $\frac{2^{s+2}-s-3}{2} \leq \frac{2^{1-s}+1}{2^{1-s}}$.

Remark 3. In Corollary 2, if we take $\alpha = \frac{1}{2}$, and $\lambda = 1$, then we get the following trapezoid inequality

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{8} \left(\frac{2^{1-s}}{(s+1)(s+2)} \right)^{\frac{1}{q}} \\ \times \left\{ (|f'(b)|^q + |f'(a)|^q (2^{s+1} + 1))^{\frac{1}{q}} + (|f'(a)|^q + |f'(b)|^q (2^{s+1} + 1))^{\frac{1}{q}} \right\}$$

Using Lemma 1 we shall give another result for convex functions as follows.

Theorem 11. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I^\circ$ with $a < b$ and $\alpha, \lambda \in [0, 1]$. If $|f'|^q$ is h -convex on $[a, b]$, $q > 1$, then the following inequality holds:

(2.12)

$$\left| \lambda(\alpha f(a) + (1-\alpha)f(b)) + (1-\lambda)f(\alpha a + (1-\alpha)b) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq (b-a) \\ \times \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\int_0^1 h(t) dt \right)^{\frac{1}{q}} \cdot \begin{cases} \left[\varepsilon_1^{\frac{1}{p}} C^{\frac{1}{q}} + \varepsilon_3^{\frac{1}{p}} D^{\frac{1}{q}} \right], & \alpha\lambda \leq 1-\alpha \leq 1-\lambda(1-\alpha) \\ \left[\varepsilon_1^{\frac{1}{p}} C^{\frac{1}{q}} + \varepsilon_4^{\frac{1}{p}} D^{\frac{1}{q}} \right], & \alpha\lambda \leq 1-\lambda(1-\alpha) \leq 1-\alpha \\ \left[\varepsilon_2^{\frac{1}{p}} C^{\frac{1}{q}} + \varepsilon_3^{\frac{1}{p}} D^{\frac{1}{q}} \right], & 1-\alpha \leq \alpha\lambda \leq 1-\lambda(1-\alpha) \end{cases}$$

where

$$(2.13) \quad \begin{aligned} C &= (1-\alpha) [|f'((1-\alpha)b + \alpha a)|^q + |f'(a)|^q], \\ D &= \alpha [|f'((1-\alpha)b + \alpha a)|^q + |f'(b)|^q], \end{aligned}$$

$$\begin{aligned} \varepsilon_1 &= (\alpha\lambda)^{p+1} + (1-\alpha-\alpha\lambda)^{p+1}, \quad \varepsilon_2 = (\alpha\lambda)^{p+1} - (\alpha\lambda - 1 + \alpha)^{p+1}, \\ \varepsilon_3 &= [\lambda(1-\alpha)]^{p+1} + [\alpha - \lambda(1-\alpha)]^{p+1}, \quad \varepsilon_4 = [\lambda(1-\alpha)]^{p+1} - [\lambda(1-\alpha) - \alpha]^{p+1}, \\ \text{and } \frac{1}{p} + \frac{1}{q} &= 1. \end{aligned}$$

Proof. From Lemma 1 and by Hölder's integral inequality, we have

$$\left| \lambda(\alpha f(a) + (1-\alpha)f(b)) + (1-\lambda)f(\alpha a + (1-\alpha)b) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ \leq (b-a) \left[\int_0^{1-\alpha} |t - \alpha\lambda| |f'(tb + (1-t)a)| dt + \int_{1-\alpha}^1 |t - 1 + \lambda(1-\alpha)| |f'(tb + (1-t)a)| dt \right] \\ \leq (b-a) \left\{ \left(\int_0^{1-\alpha} |t - \alpha\lambda|^p dt \right)^{\frac{1}{p}} \left(\int_0^{1-\alpha} |f'(tb + (1-t)a)|^q dt \right)^{\frac{1}{q}} \right.$$

$$(2.14) \quad + \left(\int_{1-\alpha}^1 |t-1+\lambda(1-\alpha)|^p dt \right)^{\frac{1}{p}} \left(\int_{1-\alpha}^1 |f'(tb+(1-t)a)|^q dt \right)^{\frac{1}{q}} \Bigg\}.$$

Since $|f'|^q$ is h -convex on $[a, b]$, for $\alpha \in [0, 1]$ by the inequality (1.4), we get

$$(2.15) \quad \int_0^{1-\alpha} |f'(tb+(1-t)a)|^q dt = (1-\alpha) \left[\frac{1}{(1-\alpha)(b-a)} \int_a^{(1-\alpha)b+\alpha a} |f'(x)|^q dx \right] \\ \leq (1-\alpha) [|f'((1-\alpha)b+\alpha a)|^q + |f'(a)|^q] \int_0^1 h(t) dt.$$

The inequality (2.15) holds for $\alpha = 1$ too. Similarly, for $\alpha \in (0, 1]$ by the inequality (1.4), we have

$$(2.16) \quad \int_{1-\alpha}^1 |f'(tb+(1-t)a)|^q dt = \alpha \left[\frac{1}{\alpha(b-a)} \int_{(1-\alpha)b+\alpha a}^b |f'(x)|^q dx \right] \\ \leq \alpha [|f'((1-\alpha)b+\alpha a)|^q + |f'(b)|^q] \int_0^1 h(t) dt.$$

The inequality (2.16) holds for $\alpha = 0$ too. By simple computation

$$(2.17) \quad \int_0^{1-\alpha} |t-\alpha\lambda|^p dt = \begin{cases} \frac{(\alpha\lambda)^{p+1} + (1-\alpha-\alpha\lambda)^{p+1}}{p+1}, & \alpha\lambda \leq 1-\alpha \\ \frac{(\alpha\lambda)^{p+1} - (\alpha\lambda-1+\alpha)^{p+1}}{p+1}, & \alpha\lambda \geq 1-\alpha \end{cases},$$

and

$$(2.18) \quad \int_{1-\alpha}^1 |t-1+\lambda(1-\alpha)|^p dt = \begin{cases} \frac{[\lambda(1-\alpha)]^{p+1} + [\alpha-\lambda(1-\alpha)]^{p+1}}{p+1}, & 1-\alpha \leq 1-\lambda(1-\alpha) \\ \frac{[\lambda(1-\alpha)]^{p+1} - [\lambda(1-\alpha)-\alpha]^{p+1}}{p+1}, & 1-\alpha \geq 1-\lambda(1-\alpha) \end{cases},$$

thus, using (2.15)-(2.18) in (2.14), we obtain the inequality (2.12). This completes the proof. \square

Corollary 5. Under the assumptions of Theorem 11 with $I \subseteq [0, \infty)$, $h(t) = t^s$, $s \in (0, 1]$, we have

$$(2.19) \quad \left| \lambda(\alpha f(a) + (1-\alpha)f(b)) + (1-\lambda)f(\alpha a + (1-\alpha)b) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq (b-a) \\ \times \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\frac{1}{s+1} \right)^{\frac{1}{q}} \cdot \begin{cases} \left[\varepsilon_1^{\frac{1}{p}} C^{\frac{1}{q}} + \varepsilon_3^{\frac{1}{p}} D^{\frac{1}{q}} \right], & \alpha\lambda \leq 1-\alpha \leq 1-\lambda(1-\alpha) \\ \left[\varepsilon_1^{\frac{1}{p}} C^{\frac{1}{q}} + \varepsilon_4^{\frac{1}{p}} D^{\frac{1}{q}} \right], & \alpha\lambda \leq 1-\lambda(1-\alpha) \leq 1-\alpha \\ \left[\varepsilon_2^{\frac{1}{p}} C^{\frac{1}{q}} + \varepsilon_3^{\frac{1}{p}} D^{\frac{1}{q}} \right], & 1-\alpha \leq \alpha\lambda \leq 1-\lambda(1-\alpha) \end{cases},$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, C$ and D are defined as in (2.13).

Corollary 6. Under the assumptions of Theorem 11 with $h(t) = t$, we have

$$\left| \lambda(\alpha f(a) + (1-\alpha)f(b)) + (1-\lambda)f(\alpha a + (1-\alpha)b) - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq (b-a) \\ \times \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\frac{1}{2} \right)^{\frac{1}{q}} \cdot \begin{cases} \left[\varepsilon_1^{\frac{1}{p}} C^{\frac{1}{q}} + \varepsilon_3^{\frac{1}{p}} D^{\frac{1}{q}} \right], & \alpha\lambda \leq 1-\alpha \leq 1-\lambda(1-\alpha) \\ \left[\varepsilon_1^{\frac{1}{p}} C^{\frac{1}{q}} + \varepsilon_4^{\frac{1}{p}} D^{\frac{1}{q}} \right], & \alpha\lambda \leq 1-\lambda(1-\alpha) \leq 1-\alpha \\ \left[\varepsilon_2^{\frac{1}{p}} C^{\frac{1}{q}} + \varepsilon_3^{\frac{1}{p}} D^{\frac{1}{q}} \right], & 1-\alpha \leq \alpha\lambda \leq 1-\lambda(1-\alpha) \end{cases},$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, C$ and D are defined as in (2.13).

Corollary 7. Under the assumptions of Theorem 11 with $h(t) = 1$, we have

$$\left| \lambda(\alpha f(a) + (1-\alpha)f(b)) + (1-\lambda)f(\alpha a + (1-\alpha)b) - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq (b-a) \\ \times \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \cdot \begin{cases} \left[\varepsilon_1^{\frac{1}{p}} C^{\frac{1}{q}} + \varepsilon_3^{\frac{1}{p}} D^{\frac{1}{q}} \right], & \alpha\lambda \leq 1-\alpha \leq 1-\lambda(1-\alpha) \\ \left[\varepsilon_1^{\frac{1}{p}} C^{\frac{1}{q}} + \varepsilon_4^{\frac{1}{p}} D^{\frac{1}{q}} \right], & \alpha\lambda \leq 1-\lambda(1-\alpha) \leq 1-\alpha \\ \left[\varepsilon_2^{\frac{1}{p}} C^{\frac{1}{q}} + \varepsilon_3^{\frac{1}{p}} D^{\frac{1}{q}} \right], & 1-\alpha \leq \alpha\lambda \leq 1-\lambda(1-\alpha) \end{cases},$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, C$ and D are defined as in (2.13).

Remark 4. In Corollary 5, if we take $\alpha = \frac{1}{2}$ and $\lambda = \frac{1}{3}$, then we have the following Simpson type inequality

$$(2.20) \quad \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right| \\ \leq \frac{b-a}{12} \left(\frac{1+2^{p+1}}{3(p+1)} \right)^{\frac{1}{p}} \left\{ \left(\frac{|f'(\frac{a+b}{2})|^q + |f'(a)|^q}{s+1} \right)^{\frac{1}{q}} + \left(\frac{|f'(\frac{a+b}{2})|^q + |f'(b)|^q}{s+1} \right)^{\frac{1}{q}} \right\},$$

which is the same of the inequality (1.8).

Remark 5. In Corollary 5, if we take $\alpha = \frac{1}{2}$ and $\lambda = 0$, then we have the following midpoint inequality

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x)dx \right| \\ \leq \frac{b-a}{4} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left\{ \left(\frac{|f'(\frac{a+b}{2})|^q + |f'(a)|^q}{s+1} \right)^{\frac{1}{q}} + \left(\frac{|f'(\frac{a+b}{2})|^q + |f'(b)|^q}{s+1} \right)^{\frac{1}{q}} \right\}.$$

We note that by inequality

$$2^{s-1} \left| f'\left(\frac{a+b}{2}\right) \right|^q \leq \frac{|f'(a)|^q + |f'(b)|^q}{s+1}$$

we have

$$\begin{aligned} \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| &\leq \left(\frac{b-a}{4}\right) \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{1}{s+1}\right)^{\frac{2}{q}} \\ &\quad \times \left[((2^{1-s} + s + 1) |f'(a)|^q + 2^{1-s} |f'(b)|^q)^{\frac{1}{q}} \right. \\ &\quad \left. + ((2^{1-s} + s + 1) |f'(b)|^q + 2^{1-s} |f'(a)|^q)^{\frac{1}{q}} \right], \end{aligned}$$

which is the same of the inequality (1.7).

Remark 6. In Corollary 5, if we take $\alpha = \frac{1}{2}$ and $\lambda = 1$, then we have the following trapezoid inequality

$$(2.21) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{4} \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \times \left\{ \left(\frac{|f'(\frac{a+b}{2})|^q + |f'(a)|^q}{s+1} \right)^{\frac{1}{q}} + \left(\frac{|f'(\frac{a+b}{2})|^q + |f'(b)|^q}{s+1} \right)^{\frac{1}{q}} \right\}.$$

We note that the obtained midpoint inequality (2.21) is better than the inequality (1.9).

Theorem 12. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I^\circ$ with $a < b$ and $\alpha, \lambda \in [0, 1]$. If $|f'|^q$ is h -concave on $[a, b]$, $q > 1$, then the following inequality holds:

$$(2.22) \quad \left| \lambda(\alpha f(a) + (1-\alpha)f(b)) + (1-\lambda)f(\alpha a + (1-\alpha)b) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq (b-a) \times \left(\frac{1}{2h(\frac{1}{2})}\right)^{\frac{1}{q}} \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \cdot \begin{cases} \left[\varepsilon_1^{\frac{1}{p}} E^{\frac{1}{q}} + \varepsilon_3^{\frac{1}{p}} F^{\frac{1}{q}} \right], & \alpha\lambda \leq 1-\alpha \leq 1-\lambda(1-\alpha) \\ \left[\varepsilon_1^{\frac{1}{p}} E^{\frac{1}{q}} + \varepsilon_4^{\frac{1}{p}} F^{\frac{1}{q}} \right], & \alpha\lambda \leq 1-\lambda(1-\alpha) \leq 1-\alpha \\ \left[\varepsilon_2^{\frac{1}{p}} E^{\frac{1}{q}} + \varepsilon_3^{\frac{1}{p}} F^{\frac{1}{q}} \right], & 1-\alpha \leq \alpha\lambda \leq 1-\lambda(1-\alpha) \end{cases}$$

where

$$E = (1-\alpha) \left| f' \left(\frac{(1-\alpha)b + (1+\alpha)a}{2} \right) \right|^q, \quad F = \alpha \left| f' \left(\frac{(2-\alpha)b + \alpha a}{2} \right) \right|^q,$$

and $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ are defined as in (2.13).

Proof. We proceed similarly as in the proof Theorem 11. Since $|f'|^q$ is h -concave on $[a, b]$, for $\alpha \in [0, 1)$ by the inequality (1.4), we get

$$(2.23) \quad \begin{aligned} \int_0^{1-\alpha} |f'(tb + (1-t)a)|^q dt &= (1-\alpha) \left[\frac{1}{(1-\alpha)(b-a)} \int_a^{(1-\alpha)b + \alpha a} |f'(x)|^q dx \right] \\ &\leq \frac{(1-\alpha)}{2h(\frac{1}{2})} \left| f' \left(\frac{(1-\alpha)b + (1+\alpha)a}{2} \right) \right|^q \end{aligned}$$

The inequality (2.23) holds for $\alpha = 1$ too. Similarly, for $\alpha \in (0, 1]$ by the inequality (1.4), we have

$$(2.24) \quad \begin{aligned} \int_{1-\alpha}^1 |f'(tb + (1-t)a)|^q dt &= \alpha \left[\frac{1}{\alpha(b-a)} \int_{(1-\alpha)b+\alpha a}^b |f'(x)|^q dx \right] \\ &\leq \frac{\alpha}{2h(\frac{1}{2})} \left| f' \left(\frac{(2-\alpha)b + \alpha a}{2} \right) \right|^q \end{aligned}$$

The inequality (2.24) holds for $\alpha = 0$ too. Thus, using (2.17), (2.18), (2.23) and (2.24) in (2.14), we obtain the inequality (2.22). This completes the proof. \square

Corollary 8. *Under the assumptions of Theorem 12 with $I \subseteq [0, \infty)$, $h(t) = t^s$, $s \in (0, 1]$, we have*

$$\begin{aligned} &\left| \lambda(\alpha f(a) + (1-\alpha)f(b)) + (1-\lambda)f(\alpha a + (1-\alpha)b) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq (b-a) \\ &\quad \times 2^{\frac{s-1}{q}} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \cdot \begin{cases} \left[\varepsilon_1^{\frac{1}{p}} E^{\frac{1}{q}} + \varepsilon_3^{\frac{1}{p}} F^{\frac{1}{q}} \right], & \alpha\lambda \leq 1-\alpha \leq 1-\lambda(1-\alpha) \\ \left[\varepsilon_1^{\frac{1}{p}} E^{\frac{1}{q}} + \varepsilon_4^{\frac{1}{p}} F^{\frac{1}{q}} \right], & \alpha\lambda \leq 1-\lambda(1-\alpha) \leq 1-\alpha \\ \left[\varepsilon_2^{\frac{1}{p}} E^{\frac{1}{q}} + \varepsilon_3^{\frac{1}{p}} F^{\frac{1}{q}} \right], & 1-\alpha \leq \alpha\lambda \leq 1-\lambda(1-\alpha) \end{cases} \end{aligned}$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, E$ and F are defined as in Theorem 12.

Corollary 9. *Under the assumptions of Theorem 12 with $h(t) = t$, we have*

$$\begin{aligned} &\left| \lambda(\alpha f(a) + (1-\alpha)f(b)) + (1-\lambda)f(\alpha a + (1-\alpha)b) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq (b-a) \\ &\quad \times \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \cdot \begin{cases} \left[\varepsilon_1^{\frac{1}{p}} E^{\frac{1}{q}} + \varepsilon_3^{\frac{1}{p}} F^{\frac{1}{q}} \right], & \alpha\lambda \leq 1-\alpha \leq 1-\lambda(1-\alpha) \\ \left[\varepsilon_1^{\frac{1}{p}} E^{\frac{1}{q}} + \varepsilon_4^{\frac{1}{p}} F^{\frac{1}{q}} \right], & \alpha\lambda \leq 1-\lambda(1-\alpha) \leq 1-\alpha \\ \left[\varepsilon_2^{\frac{1}{p}} E^{\frac{1}{q}} + \varepsilon_3^{\frac{1}{p}} F^{\frac{1}{q}} \right], & 1-\alpha \leq \alpha\lambda \leq 1-\lambda(1-\alpha) \end{cases} \end{aligned}$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, E$ and F are defined as in Theorem 12.

Corollary 10. *Under the assumptions of Theorem 12 with $h(t) = 1$, we have*

$$\begin{aligned} &\left| \lambda(\alpha f(a) + (1-\alpha)f(b)) + (1-\lambda)f(\alpha a + (1-\alpha)b) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq (b-a) \\ &\quad \times 2^{-\frac{1}{q}} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \cdot \begin{cases} \left[\varepsilon_1^{\frac{1}{p}} E^{\frac{1}{q}} + \varepsilon_3^{\frac{1}{p}} F^{\frac{1}{q}} \right], & \alpha\lambda \leq 1-\alpha \leq 1-\lambda(1-\alpha) \\ \left[\varepsilon_1^{\frac{1}{p}} E^{\frac{1}{q}} + \varepsilon_4^{\frac{1}{p}} F^{\frac{1}{q}} \right], & \alpha\lambda \leq 1-\lambda(1-\alpha) \leq 1-\alpha \\ \left[\varepsilon_2^{\frac{1}{p}} E^{\frac{1}{q}} + \varepsilon_3^{\frac{1}{p}} F^{\frac{1}{q}} \right], & 1-\alpha \leq \alpha\lambda \leq 1-\lambda(1-\alpha) \end{cases} \end{aligned}$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, E$ and F are defined as in Theorem 12.

Corollary 11. Under the assumptions of Theorem 12 with $h(t) = \frac{1}{t}$, $t \in (0, 1)$, we have

$$\left| \lambda (f' \alpha f(a) + (1 - \alpha) f(b)) + (1 - \lambda) f(\alpha a + (1 - \alpha) b) - \frac{1}{b - a} \int_a^b f(x) dx \right| \leq (b - a) \times 4^{-\frac{1}{q}} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \cdot \begin{cases} \left[\varepsilon_1^{\frac{1}{p}} E^{\frac{1}{q}} + \varepsilon_3^{\frac{1}{p}} F^{\frac{1}{q}} \right], & \alpha \lambda \leq 1 - \alpha \leq 1 - \lambda(1 - \alpha) \\ \left[\varepsilon_1^{\frac{1}{p}} E^{\frac{1}{q}} + \varepsilon_4^{\frac{1}{p}} F^{\frac{1}{q}} \right], & \alpha \lambda \leq 1 - \lambda(1 - \alpha) \leq 1 - \alpha \\ \left[\varepsilon_2^{\frac{1}{p}} E^{\frac{1}{q}} + \varepsilon_3^{\frac{1}{p}} F^{\frac{1}{q}} \right], & 1 - \alpha \leq \alpha \lambda \leq 1 - \lambda(1 - \alpha) \end{cases},$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, E$ and F are defined as in Theorem 12.

Remark 7. In Corollary 8, if we take $\alpha = \frac{1}{2}$ and $\lambda = 1$, then we have the following trapezoid inequality

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) dx \right| \\ & \leq \frac{b - a}{4} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \times \left(\frac{1}{2} \right)^{\frac{1-s}{q}} \left[\left| f' \left(\frac{3b+a}{4} \right) \right| + \left| f' \left(\frac{3a+b}{4} \right) \right| \right] \end{aligned}$$

which is the same of the inequality in [16, Theorem 8 (i)].

Remark 8. In Corollary 8, if we take $\alpha = \frac{1}{2}$ and $\lambda = 0$, then we have the following midpoint inequality

$$\begin{aligned} & \left| f \left(\frac{a+b}{2} \right) - \frac{1}{b - a} \int_a^b f(x) dx \right| \\ & \leq \frac{b - a}{4} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \times \left(\frac{1}{2} \right)^{\frac{1-s}{q}} \left[\left| f' \left(\frac{3b+a}{4} \right) \right| + \left| f' \left(\frac{3a+b}{4} \right) \right| \right] \end{aligned}$$

which is the same of the inequality in [16, Theorem 8 (ii)].

Remark 9. In Corollary 9, if we take $\alpha = \frac{1}{2}$ and $\lambda = 1$, then we have the following trapezoid inequality

$$\begin{aligned} (2.25) \quad & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) dx \right| \\ & \leq \frac{b - a}{4} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left[\left| f' \left(\frac{3b+a}{4} \right) \right| + \left| f' \left(\frac{3a+b}{4} \right) \right| \right] \end{aligned}$$

which is the same of the inequality in [12, Theorem 2].

Remark 10. In Corollary 9, if we take $\alpha = \frac{1}{2}$ and $\lambda = 0$, then we have the following trapezoid inequality

$$(2.26) \quad \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{4} \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left[\left| f'\left(\frac{3b+a}{4}\right) \right| + \left| f'\left(\frac{3a+b}{4}\right) \right| \right]$$

which is the same of the inequality in [2, Theorem 2.5].

Remark 11. In Corollary 9, since $|f'|^q$, $q > 1$, is concave on $[a, b]$, using the power mean inequality, we have

$$\begin{aligned} |f'(\lambda x + (1-\lambda)y)|^q &\geq \lambda |f'(x)|^q + (1-\lambda) |f'(y)|^q \\ &\geq (\lambda |f'(x)| + (1-\lambda) |f'(y)|)^q, \end{aligned}$$

$\forall x, y \in [a, b]$ and $\lambda \in [0, 1]$. Hence

$$|f'(\lambda x + (1-\lambda)y)| \geq \lambda |f'(x)| + (1-\lambda) |f'(y)|$$

so $|f'|$ is also concave. Then by the inequality (1.1), we have

$$(2.27) \quad \left| f'\left(\frac{3b+a}{4}\right) \right| + \left| f'\left(\frac{3a+b}{4}\right) \right| \leq 2 \left| f'\left(\frac{a+b}{2}\right) \right|.$$

Thus, using the inequality (2.27) in (2.25) and (2.26) we get

$$\begin{aligned} \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| &\leq \frac{b-a}{2} \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left| f'\left(\frac{a+b}{2}\right) \right|, \\ \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x)dx \right| &\leq \frac{b-a}{2} \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left| f'\left(\frac{a+b}{2}\right) \right|. \end{aligned}$$

3. SOME APPLICATIONS FOR SPECIAL MEANS

Let us recall the following special means of arbitrary real numbers a, b with $a \neq b$ and $\alpha \in [0, 1]$:

(1) The weighted arithmetic mean

$$A_\alpha(a, b) := \alpha a + (1-\alpha)b, \quad a, b \in \mathbb{R}.$$

(2) The unweighted arithmetic mean

$$A(a, b) := \frac{a+b}{2}, \quad a, b \in \mathbb{R}.$$

(3) The Logarithmic mean

$$L(a, b) := \frac{b-a}{\ln|b| - \ln|a|}, \quad |a| \neq |b|, \quad ab \neq 0.$$

(4) Then p -Logarithmic mean

$$L_p(a, b) := \left(\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right)^{\frac{1}{p}}, \quad p \in \mathbb{R} \setminus \{-1, 0\}, \quad a, b > 0.$$

From known Example 1 in [9], we may find that for any $s \in (0, 1)$ and $\beta > 0$, $f : [0, \infty) \rightarrow [0, \infty)$, $f(t) = \beta t^s$, $f \in K_s^2$.

Now, using the results of Section 2, some new inequalities are derived for the above means.

Proposition 1. *Let $a, b \in \mathbb{R}$ with $0 < a < b$, $q \geq 1$ and $s \in \left(0, \frac{1}{q}\right)$ we have the following inequality:*

$$\begin{aligned} & \left| \lambda A_\alpha(a^{s+1}, b^{s+1}) + (1 - \lambda) A_\alpha^{s+1}(a, b) - L_{s+1}^{s+1}(a, b) \right| \\ & \leq \begin{cases} (b-a)(s+1) \left\{ \gamma_2^{1-\frac{1}{q}} (\mu_1^* b^{sq} + \mu_2^* a^{sq})^{\frac{1}{q}} \right. \\ \quad \left. + v_2^{1-\frac{1}{q}} (\eta_3^* b^{sq} + \eta_4^* a^{sq})^{\frac{1}{q}} \right\}, & \alpha\lambda \leq 1 - \alpha \leq 1 - \lambda(1 - \alpha) \\ (b-a)(s+1) \left\{ \gamma_2^{1-\frac{1}{q}} (\mu_1^* b^{sq} + \mu_2^* a^{sq})^{\frac{1}{q}} \right. \\ \quad \left. + v_1^{1-\frac{1}{q}} (\eta_1^* b^{sq} + \eta_2^* a^{sq})^{\frac{1}{q}} \right\}, & \alpha\lambda \leq 1 - \lambda(1 - \alpha) \leq 1 - \alpha, \\ (b-a)(s+1) \left\{ \gamma_1^{1-\frac{1}{q}} (\mu_3^* b^{sq} + \mu_4^* a^{sq})^{\frac{1}{q}} \right. \\ \quad \left. + v_2^{1-\frac{1}{q}} (\eta_3^* b^{sq} + \eta_4^* a^{sq})^{\frac{1}{q}} \right\}, & 1 - \alpha \leq \alpha\lambda \leq 1 - \lambda(1 - \alpha) \end{cases}, \end{aligned}$$

where $\gamma_1, \gamma_2, v_1, v_2, \mu_1^*, \mu_2^*, \mu_3^*, \mu_4^*, \eta_1^*, \eta_2^*, \eta_3^*, \eta_4^*$ numbers are defined as in Corollary 2.

Proof. The assertion follows from applied the inequality (2.9) to the function $f(t) = t^{s+1}$, $t \in [a, b]$ and $s \in \left(0, \frac{1}{q}\right)$, which implies that $f'(t) = (s+1)t^s$, $t \in [a, b]$ and $|f'(t)|^q = (s+1)^q t^{qs}$, $t \in [a, b]$ is a s -convex function in the second sense since $qs \in (0, 1)$ and $(s+1)^q > 0$. \square

Proposition 2. *Let $a, b \in \mathbb{R}$ with $0 < a < b$, $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $s \in \left(0, \frac{1}{q}\right)$ we have the following inequality:*

$$\begin{aligned} & \left| \lambda A_\alpha(a^{s+1}, b^{s+1}) + (1 - \lambda) A_\alpha^{s+1}(a, b) - L_{s+1}^{s+1}(a, b) \right| \leq (b-a) \left(\frac{1}{p+1} \right)^{\frac{1}{p}} (s+1)^{1-\frac{1}{q}} \\ & \times \begin{cases} \left[(1-\alpha)^{\frac{1}{q}} \varepsilon_1^{\frac{1}{p}} \theta_1 + \alpha^{\frac{1}{q}} \varepsilon_3^{\frac{1}{p}} \theta_2 \right], & \alpha\lambda \leq 1 - \alpha \leq 1 - \lambda(1 - \alpha) \\ \left[(1-\alpha)^{\frac{1}{q}} \varepsilon_1^{\frac{1}{p}} \theta_1 + \alpha^{\frac{1}{q}} \varepsilon_4^{\frac{1}{p}} \theta_2 \right], & \alpha\lambda \leq 1 - \lambda(1 - \alpha) \leq 1 - \alpha, \\ \left[(1-\alpha)^{\frac{1}{q}} \varepsilon_2^{\frac{1}{p}} \theta_1 + \alpha^{\frac{1}{q}} \varepsilon_3^{\frac{1}{p}} \theta_2 \right], & 1 - \alpha \leq \alpha\lambda \leq 1 - \lambda(1 - \alpha) \end{cases} \end{aligned}$$

where

$$\theta_1 = A_\alpha^{sq}(a, b) + a^{sq}, \quad \theta_2 = A_\alpha^{sq}(a, b) + b^{sq},$$

and $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ numbers are defined as in Corollary 5.

Proof. The assertion follows from applied the inequality (2.19) to the function $f(t) = t^{s+1}$, $t \in [a, b]$ and $s \in \left(0, \frac{1}{q}\right)$, which implies that $f'(t) = (s+1)t^s$, $t \in [a, b]$ and $|f'(t)|^q = (s+1)^q t^{qs}$, $t \in [a, b]$ is a s -convex function in the second sense since $qs \in (0, 1)$ and $(s+1)^q > 0$. \square

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